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THE 2-RAY ALGORITHM FOR SOLVING EQUILIBRIUM PROBLEMS ON THE UNIT SIMPLEX

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ABSTRACT

In this paper we present a simplicial variable dimension restart algorithm to find economic equilibria on the n -dimensional unit simplex S^n with just two rays to leave the arbitrarily chosen starting point. From this point the algorithm generates a sequence of adjacent simplices of varying dimension in some simplicial subdivision of S^n until a simplex is found which yields an approximate solution. In this way the algorithm traces a piecewise linear path of points which can be considered as an approximation of the piecewise smooth path followed by some price adjustment process. This process solves a sequence of subproblems of varying dimension.

1. Introduction

In this paper a new simplicial variable dimension restart algorithm is presented to solve the equilibrium problem

$$z(x) = 0 \quad x \in S^n \quad (1.1)$$

where z is a continuous function from the n -dimensional unit simplex $S^n = \{x \in R_+^{n+1} \mid \sum_{i=1}^{n+1} x_i = 1\}$ into R^{n+1} satisfying the conditions $x^T z(x) = 0$ for all x in S^n and, for all $i=1, \dots, n+1$, $z_i(x) > 0$ if $x_i = 0$. Simplicial algorithms differ from each other in the number of rays along which the algorithm can leave the arbitrarily chosen starting point. From this point such an algorithm generates a sequence of

adjacent simplices of varying dimension of a simplicial subdivision of S^n until an approximating simplex is found. An algorithm with $n+1$ rays was developed in van der Laan and Talman (4) (see also Doup and Talman (1)) and with $2^{n+1}-2$ rays in Doup, van der Laan and Talman (2).

Simplicial algorithms to solve the zero point problem on R^n were developed with $n+1$ and $2n$ rays in van der Laan and Talman (5), with 2^n rays in Wright (8) and with 3^n-1 rays in Kojima and Yamamoto (3). These algorithms on R^n are closely related to the algorithms on S^n mentioned above. Moreover, a simplicial algorithm on R^n exists with only two rays. This algorithm, introduced in Saigal (7) and independently in Yamamoto (9), solves a sequence of subproblems of varying dimension.

We now propose a 2-ray algorithm on S^n which solves for varying t , $1 \leq t \leq n$, a sequence of subproblems consisting of the first t equations of (1.1). The dimension t , however, needs not to increase monotonically from 1 to n .

In section 2 we describe the piecewise smooth path of points in S^n of the adjustment process which is approximately followed by the 2-ray algorithm. In section 3 the steps of the algorithm to follow the approximating piecewise linear path are presented. This path is traced by the algorithm by alternating linear programming pivot steps and replacement steps in some underlying simplicial subdivision of S^n . In section 4 some concluding remarks are given.

2. The 2-ray process

The equilibrium problem (EP) can be stated as follows. Let z be a continuous function from S^n into R^{n+1} such that $x^T z(x) = 0$ for all x in S^n and for all i in the index set $I_{n+1} = \{1, \dots, n+1\}$, $x_i = 0$ implies $z_i(x) > 0$, then the problem is to find an x^* in S^n such that $z(x^*) = 0$. The first condition is the so-called Walras' law and the second condition gives us that x^* lies in the interior of S^n .

Let $x = v$ be an interior starting point in S^n , then the 2-ray process starts by viewing the sign of only $z_1(v)$. If $z_1(v)$ is positive, then x_1 is increased away from v_1 and all other components of x are kept relatively to v equal to each other and lower than x_1/v_1 . If $z_1(v)$ is negative, then x_1 is decreased away from v_1 and all other components of x are kept relatively to v equal to each other and larger than x_1/v_1 . The first region is described by

$$A(1) = \{x \in S^n \mid x_2/v_2 = \dots = x_{n+1}/v_{n+1} < x_1/v_1\}$$

whereas the second region is described by

$$A(-1) = \{x \in S^n \mid x_2/v_2 = \dots = x_{n+1}/v_{n+1} > x_1/v_1\}.$$

The idea behind the 2-ray process is to solve a sequence of problems of varying dimension, not necessarily monotonic. We start by solving the problem $z_1(x) = 0$ and successively $z_1(x) = 0, \dots, z_t(x) = 0$ for $t=2, \dots, n$. By Walras' law this implies that a solution x for $t=n$ also satisfies $z_{n+1}(x) = 0$, i.e. x is a solution to the EP.

In general, the region $A(k)$, $k=t, -t$, for $t=1, \dots, n$, is given by

$$A(t) = \{x \in S^n \mid x_i/v_i = b, i=t+1, \dots, n+1, x_t/v_t > b, b > 0\}$$

and

$$A(-t) = \{x \in S^n \mid x_i/v_i = b, i=t+1, \dots, n+1, x_t/v_t < b, b > 0\}.$$

Furthermore let $A(0)$ be equal to $\{v\}$. In $A(t)$ ($A(-t)$) the process will generate points x such that $z_1(x) = 0, \dots, z_{t-1}(x) = 0$ and $z_t(x) > 0$ ($z_t(x) < 0$). Therefore, let $C(k)$, $k=t, -t$, $1 \leq t \leq n$, be given by

$$C(t) = \text{Cl}(\{x \in S^n \mid z_1(x) = 0, \dots, z_{t-1}(x) = 0, z_t(x) > 0\})$$

and

$$C(-t) = Cl(\{x \in S^n \mid z_1(x) = 0, \dots, z_{t-1}(x) = 0, z_t(x) < 0\})$$

and $C(n+1) = C(-(n+1)) = \{x \in S^n \mid z(x) = 0\}$. Furthermore let, for $k=t, -t$, $1 < t < n$, $B(k)$ be the intersection of $A(k)$ and $C(k)$. In general, if z is continuously differentiable, each $B(k)$, $k=t, -t$, consists of smooth loops and curves (see e.g. (6)) each curve having two end points in $bd B(k)$, i.e. in either $(bd A(k)) \cap C(k)$ or in $A(k) \cap (bd C(k))$. More precisely, the boundary of $B(t)$, $1 < t < n$, is equal to

$$\begin{aligned} bd B(t) = & \left(\bigcup_{k=-(t-1), (t-1)} (A(k) \cap C(t)) \right) \cup \\ & \left(\bigcup_{k=-(t+1), (t+1)} (A(t) \cap C(k)) \right) \end{aligned}$$

and the boundary of $B(-t)$, $1 < t < n$, is equal to

$$\begin{aligned} bd B(-t) = & \left(\bigcup_{k=-(t-1), (t-1)} (A(k) \cap C(-t)) \right) \cup \\ & \left(\bigcup_{k=-(t+1), (t+1)} (A(-t) \cap C(k)) \right). \end{aligned}$$

We observe that no points in $B(k)$, $k=t, -t$, lie in the boundary of S^n . This is caused by the fact that $x_i = 0$ implies $z_i(x) > 0$ and for all x in S^n Walras' law holds.

Therefore, if an end point x of a curve in $B(t)$ lies in $(bd A(t)) \cap C(t)$, then x is equal to v if $t=1$ and in the case $t > 1$ x is an end point of a curve in $B(-(t-1))$ if $x_{t-1}/v_{t-1} < b$ and in $B(t-1)$ if $x_{t-1}/v_{t-1} > b$. If the point x lies in $A(t) \cap bd C(t)$, then x is a solution of the EP if $t=n$ and in the case $t < n$ the point x is an end point of a curve in $B(-(t+1))$ if $z_{t+1}(x) < 0$ and in $B(t+1)$ if $z_{t+1}(x) > 0$. Similarly, if an end point x of a curve in

$B(-t)$ lies in $(\text{bd } A(-t)) \cap C(-t)$ then x is equal to v if $t=1$ and in the case $t > 1$, x is an end point of a curve in $B(-(t-1))$ if $x_{t-1}/v_{t-1} < b$ and in $B(t-1)$ if $x_{t-1}/v_{t-1} > b$. If the point x lies in $A(-t) \cap \text{bd } C(-t)$ then x is a solution of the EP if $t=n$, and in the case $t < n$ the point x is an end point of a curve in $B(-(t+1))$ if $z_{t+1}(x) < 0$ and in $B(t+1)$ if $z_{t+1}(x) > 0$. In this way we can link all the curves in $B(k)$, $k=t, -t, 1 < t < n$, together.

Let B be given by

$$B = \bigcup_{k \in I_{\pm n}} B(k)$$

where $I_{\pm n} = \{-n, -n+1, \dots, -1, 1, \dots, n\}$, then the set B is in general the union of piecewise smooth curves and loops and contains one curve, say P , connecting v and a solution point x^* of the EP. All other curves connect two solution points. This is illustrated in figure 1 for $n=2$.

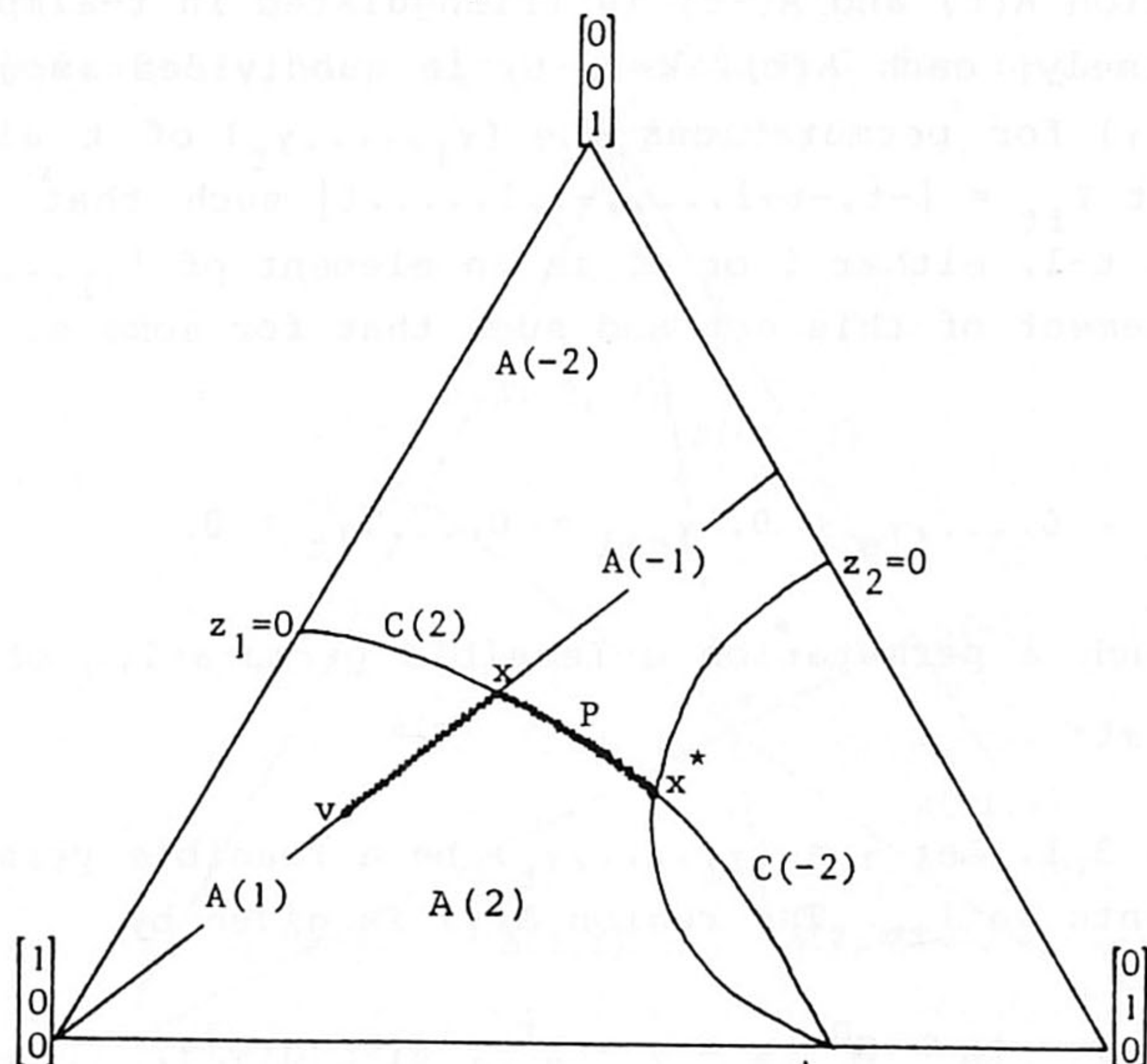


Figure 1. The curve P connects v and x^* . The sign of $z_1(v)$ is negative and in the point x we have $z_1(x) = 0$ and $z_2(x) > 0$.

The curve P will be followed approximately by the 2-ray algorithm which we will present in the following section.

3. The 2-ray algorithm on S^n

To follow the piecewise smooth curve P of the 2-ray process, this curve is approximated by the piecewise linear path generated by the steps of a simplicial variable dimension restart algorithm. We call this algorithm the 2-ray algorithm on S^n . The piecewise linear path traced by the algorithm is followed by generating a sequence of adjacent simplices of varying dimension of some simplicial subdivision of S^n containing this p.l. path. This sequence of simplices is generated by alternating pivot steps in a system of linear equations and replacement steps in the simplicial subdivision. The underlying simplicial subdivision is the so-called V-triangulation, developed in (1). Each t -dimensional region $A(t)$ and $A(-t)$ is triangulated in t -simplices. More precisely, each $A(k)$, $k=t, -t$, is subdivided in $\frac{1}{2}(t+1)!$ subsets $A(\gamma)$ for permutations $\gamma = (\gamma_1, \dots, \gamma_t)$ of t elements of the set $I_{\pm t} = \{-t, -t+1, \dots, -1, 1, \dots, t\}$ such that for all i , $1 \leq i \leq t-1$, either i or $-i$ is an element of $\{\gamma_1, \dots, \gamma_t\}$, k is an element of this set and such that for some s , $0 \leq s \leq t$,

$$\gamma_1 < 0, \dots, \gamma_s < 0, \gamma_{s+1} > 0, \dots, \gamma_t > 0.$$

We call such a permutation a feasible permutation of t elements in $I_{\pm t}$.

Definition 3.1. Let $\gamma = (\gamma_1, \dots, \gamma_t)$ be a feasible permutation of t elements in $I_{\pm t}$. The region $A(\gamma)$ is given by

$$A(\gamma) = \{x \in S^n \mid x = v + \sum_{h=1}^t \alpha(\gamma_h) q(\gamma_h),$$

$$0 < \alpha(\gamma_t) < \dots < \alpha(\gamma_1) < 1\} \quad (3.1)$$

where the vectors $q(\gamma_h)$, $h=1, \dots, t$ are given by

$$q(\gamma_h) = p(\{\gamma_1, \dots, \gamma_h\}) - p(\{\gamma_1, \dots, \gamma_{h-1}\})$$

with for $K \subset I_{-n} = \{-n, \dots, -1\}$, $K \neq \emptyset$, $p(K)$ given by

$$p_i(K) = \begin{cases} v_i / \sum_{k \notin -K} v_k & i \notin -K \\ 0 & i \in -K \end{cases}, \quad i \in I_{n+1}$$

and for $K \subset I_{\pm n}$, $K \not\subset I_{-n}$, $K \neq \emptyset$, $p(K)$ given by

$$p_i(K) = \begin{cases} v_i / \sum_{k \in K^+} v_k & i \in K^+ \\ 0 & i \notin K^+ \end{cases}, \quad i \in I_{n+1}$$

where $K^+ = \{k \in K \mid k > 0\}$. In the case $K = \emptyset$, $p(K)$ is given by $p(K) = v$.

Some regions $A(\gamma)$ are given in figure 2 for $n=2$.

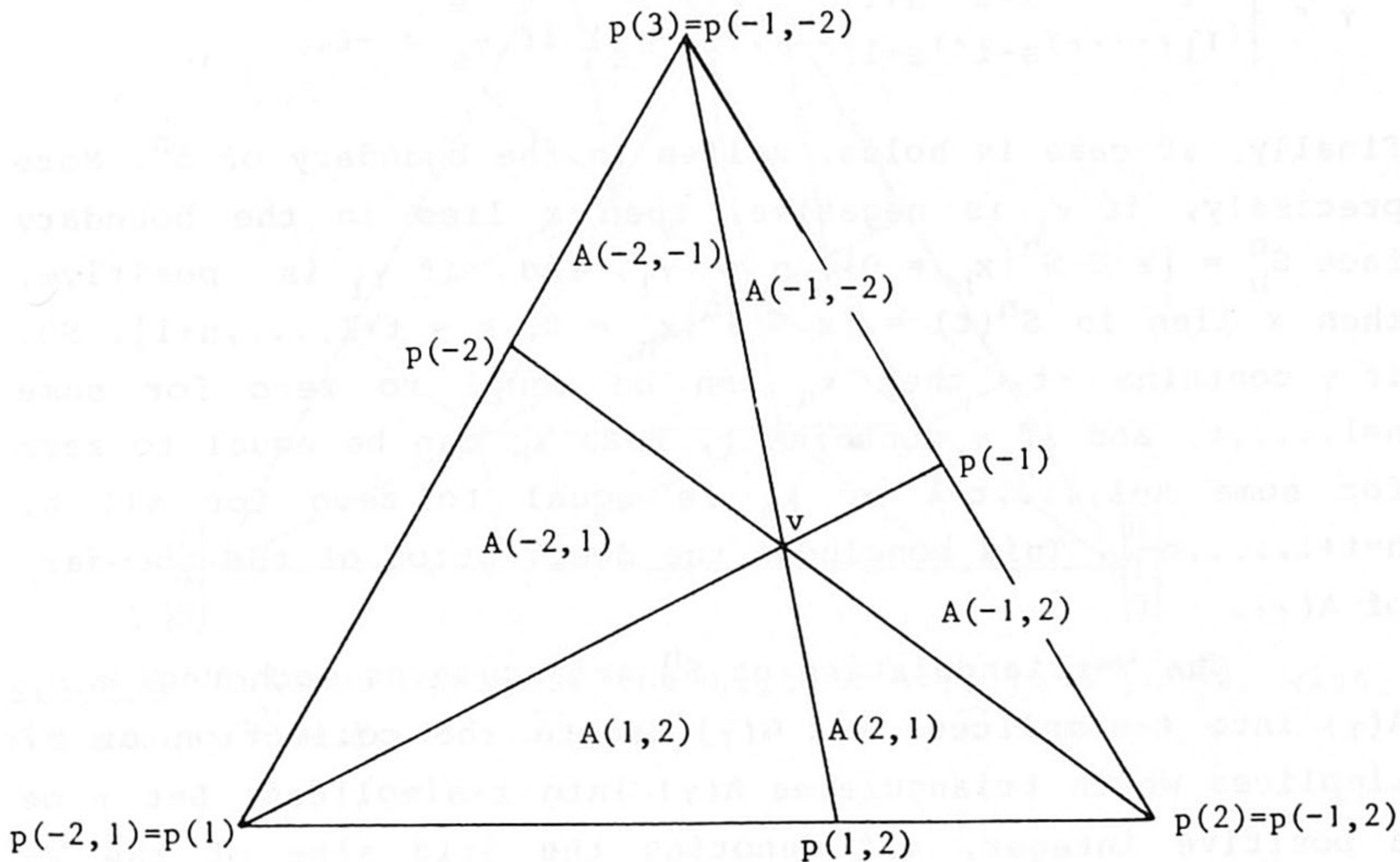


Figure 2. The 2-dimensional unit simplex subdivided into 2-dimensional regions $A(\gamma)$

The region $A(\gamma)$, $\gamma = (\gamma_1, \dots, \gamma_t)$, is t -dimensional for feasible γ . The set $A(k)$, $k=t, -t$, is equal to the union of the $A(\gamma)$'s over all $\gamma = (\gamma_1, \dots, \gamma_t)$ containing k .

The boundary of $A(\gamma)$ consists of a number of $(t-1)$ -dimensional subsets. These are obtained by setting one inequality in (3.1) to an equality. The following four cases can occur: i) $\alpha(\gamma_t) = 0$, ii) $\alpha(\gamma_i) = \alpha(\gamma_{i-1})$, $i \neq s+1$, $2 \leq i \leq t$, iii) $\alpha(\gamma_{s+1}) = \alpha(\gamma_s)$ and iv) $\alpha(\gamma_1) = 1$. Let x be an interior point in $\text{bd } A(\gamma)$. In case i, x lies also in $A(\bar{\gamma})$, with $\bar{\gamma}$ given by

$$\bar{\gamma} = \begin{cases} (\gamma_1, \dots, \gamma_{t-1}) & \text{if } \gamma_t = t \text{ or } \gamma_t = -t \\ (\gamma_1, \dots, \gamma_{t-1}, -\gamma_t) & \text{if } \gamma_t < 0 \text{ and } \gamma_t \neq -t \\ (\gamma_1, \dots, \gamma_s, -\gamma_t, \gamma_{s+1}, \dots, \gamma_{t-1}) & \text{if } \gamma_t > 0 \text{ and } \gamma_t \neq t. \end{cases}$$

In case ii, x lies also in $A(\bar{\gamma})$, with $\bar{\gamma}$ given by

$\bar{\gamma} = (\gamma_1, \dots, \gamma_{i-2}, \gamma_i, \gamma_{i-1}, \dots, \gamma_t)$. In case iii, which can only occur if $0 < s < t$, x lies also in $A(\bar{\gamma})$, with $\bar{\gamma}$ given by

$$\bar{\gamma} = \begin{cases} (\gamma_1, \dots, \gamma_{s-1}, \gamma_{s+1}, \dots, \gamma_t) & \text{if } \gamma_s = -t \\ (\gamma_1, \dots, \gamma_{s-1}, \gamma_{s+1}, \dots, \gamma_t, -\gamma_s) & \text{if } \gamma_s \neq -t. \end{cases}$$

Finally, if case iv holds, x lies in the boundary of S^n . More precisely, if γ_1 is negative, then x lies in the boundary face $S_h^n = \{x \in S^n \mid x_h = 0\}$, $h = -\gamma_1$, and if γ_1 is positive, then x lies in $S^n(t) = \{x \in S^n \mid x_h = 0, h = t+1, \dots, n+1\}$. So, if γ contains $-t$, then x_h can be equal to zero for some $h=1, \dots, t$, and if γ contains t , then x_h can be equal to zero for some $h=1, \dots, t-1$ or x_h is equal to zero for all $h, h=t+1, \dots, n+1$. This concludes the description of the boundary of $A(\gamma)$.

The V -triangulation of S^n triangulates each region $A(\gamma)$ into t -simplices. Let $G(\gamma)$ denote the collection of t -simplices which triangulates $A(\gamma)$ into t -simplices. Let m be a positive integer, m^{-1} denoting the grid size of the V -triangulation, then the collection is defined as follows.

Definition 3.2. The collection $G(\gamma)$ of t -dimensional simplices $\sigma(y^1, \pi(t))$ with vertices y^1, \dots, y^{t+1} is given by

- 1) $y^1 = v + \sum_{i=1}^t a(\gamma_i) m^{-1} q(\gamma_i)$ for integers $a(\gamma_i)$, $i=1, \dots, t$ such that $0 < a(\gamma_t) < \dots < a(\gamma_1) < m-1$
- 2) $\pi(t) = (\pi_1, \dots, \pi_t)$ is a permutation of the t elements in $\{\gamma_1, \dots, \gamma_t\}$ such that $p < p'$ if $\pi_p = \gamma_{i-1}, \pi_{p'} = \gamma_i$ for certain i and $a(\pi_p) = a(\pi_{p'})$

and

- 3) $y^{i+1} = y^i + m^{-1} q(\pi_i)$, $i=1, \dots, t$.

The triangulation of the regions $A(\gamma)$ is illustrated in figure 3 for $n=2$ and $m=2$. The arrows give the directions of the vectors $q(\cdot)$.

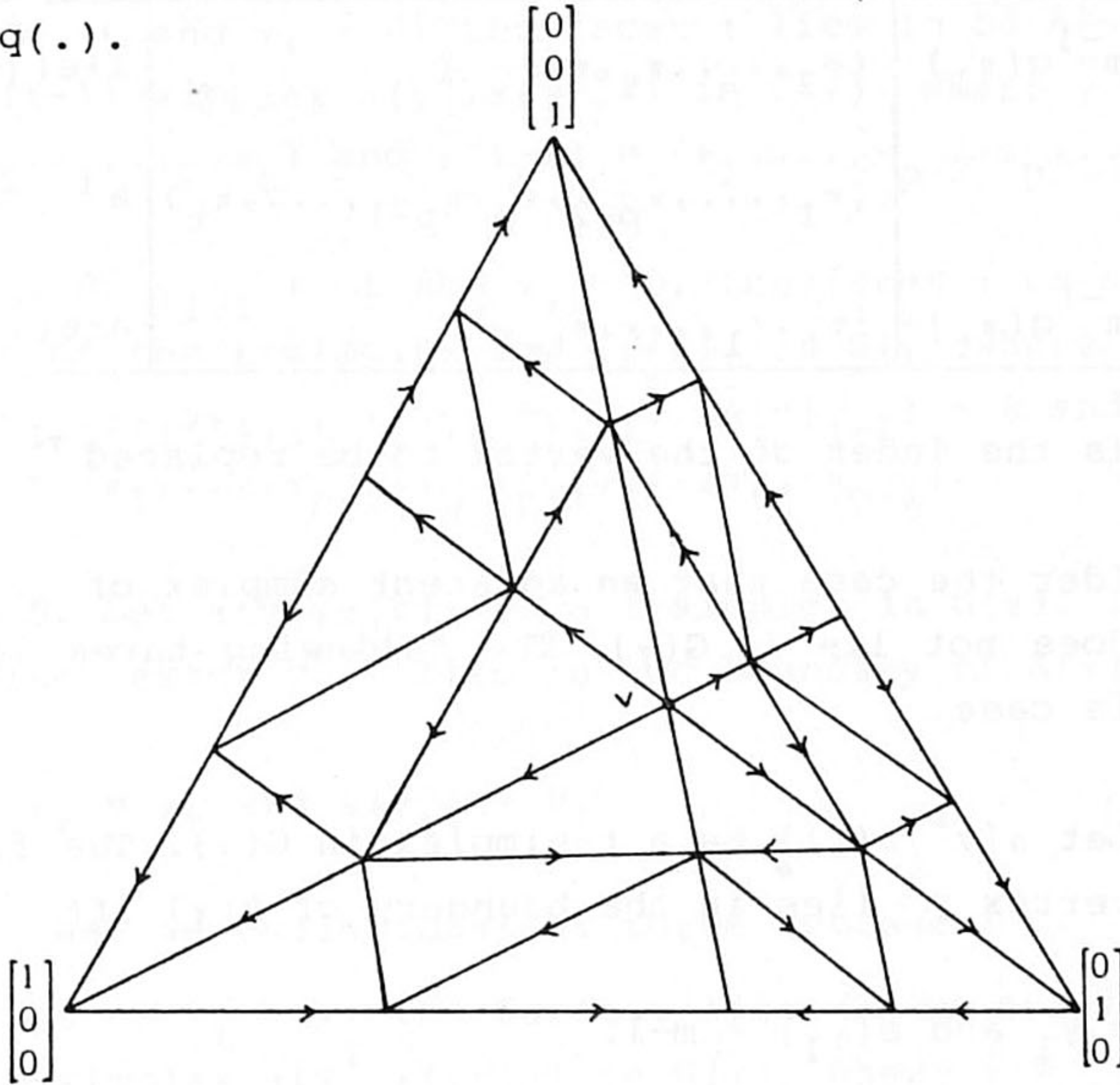


Figure 3. Triangulation of the regions $A(\gamma)$ in S^n , $n=2$, with grid size $m^{-1} = \frac{1}{2}$

We call the union of $G(\gamma)$, $\gamma = (\gamma_1, \dots, \gamma_t)$, over all γ 's such that γ contains k , the triangulation $G(k)$ of $A(k)$, $k=t, -t$, $1 \leq t \leq n$.

The algorithm in general moves from one t -simplex $\sigma(y^1, \pi(t))$ in $A(k)$, $k=t, -t$, to an adjacent t -simplex $\bar{\sigma}(\bar{y}^1, \bar{\pi}(t))$ in $A(k)$. If both simplices lie in $G(\gamma)$, the parameters $\bar{y}^1, \bar{\pi}(t)$ and \bar{a} are obtained from table 1 where p is the index of the vertex of σ such that σ and $\bar{\sigma}$ share the common facet $\tau(y^1, \dots, y^{p-1}, y^{p+1}, \dots, y^{t+1})$, where $a_j = a(\gamma_j)$, $j = |\gamma_i|$, $i=1, \dots, t$, $a_j=0$ for $j = t+1, \dots, n+1$, and where $e(h)$ is the h -th unit vector in R^{n+1} .

	\bar{y}^1	$\bar{\pi}(t)$	\bar{a}
$p=1$	$y^{1+m-1}q(\pi_1)$	$(\pi_2, \dots, \pi_t, \pi_1)$	$a+e(\pi_1)$
$1 < p < t+1$	y^1	$(\pi_1, \dots, \pi_{p-2}, \pi_p, \pi_{p-1}, \dots, \pi_t)$	a
$p=t+1$	$y^{1-m-1}q(\pi_t)$	$(\pi_t, \pi_1, \dots, \pi_{t-1})$	$a-e(\pi_t)$

Table 1. p is the index of the vertex to be replaced

We now consider the case that an adjacent simplex of $\sigma(y^1, \pi(t))$ does not lie in $G(\gamma)$. The following three lemmas describe this case.

Lemma 3.3. Let $\sigma(y^1, \pi(t))$ be a t -simplex in $G(\gamma)$. The facet τ opposite vertex y^1 lies in the boundary of $A(\gamma)$ iff

$$\pi_1 = \gamma_1 \text{ and } a(\gamma_1) = m-1.$$

In the case $\gamma_1 < 0$, τ is a $(t-1)$ -simplex in $S_{-\gamma_1}^n$, i.e. all points x in τ satisfy $x_{-\gamma_1} = 0$. In the case $\gamma_1 > 0$, τ is a $(t-1)$ -simplex in $S^n(t)$.

Lemma 3.4. Let $\sigma(y^1, \pi(t))$ be a t -simplex in $G(\gamma)$. The facet τ opposite vertex y^p , for some p , $1 < p < t+1$, lies in the boundary of $A(\gamma)$ iff

$$\pi_{p-1} = \gamma_{i-1}, \pi_p = \gamma_i \text{ for certain } i, 1 < i < t, \text{ and} \\ a(\gamma_{i-1}) = a(\gamma_i).$$

In this case we consider three subcases:

- 1) $\text{sgn}(\gamma_{i-1}) = \text{sgn}(\gamma_i)$: the facet τ is also a facet of the t -simplex $\bar{\sigma}(y^1, \bar{\pi}(t))$ in $G(\bar{\gamma})$, where $\bar{\gamma} = (\gamma_1, \dots, \gamma_{i-2}, \gamma_i, \gamma_{i-1}, \gamma_{i+1}, \dots, \gamma_t)$ and $\bar{\pi}(t) = (\pi_1, \dots, \pi_{p-2}, \pi_p, \pi_{p-1}, \pi_{p+1}, \dots, \pi_t)$
- 2) $\gamma_{i-1} = -t$ and $\gamma_i > 0$: the facet τ lies in $\text{bd } A(-t)$ and is the $(t-1)$ -simplex $\bar{\sigma}(y^1, \bar{\pi}(t-1))$ in $G(\bar{\gamma})$, where $\bar{\gamma} = (\gamma_1, \dots, \gamma_{i-2}, \gamma_i, \dots, \gamma_t)$ and $\bar{\pi}(t-1) = (\pi_1, \dots, \pi_{p-2}, \pi_p, \dots, \pi_t)$
- 3) $\gamma_{i-1} < 0$, $\gamma_{i-1} \neq -t$ and $\gamma_i > 0$: the facet τ is also a facet of the t -simplex $\bar{\sigma}(y^1, \bar{\pi}(t))$ in $G(\bar{\gamma})$ where $\bar{\gamma} = (\gamma_1, \dots, \gamma_{i-2}, \gamma_i, \gamma_{i+1}, \dots, \gamma_t, -\gamma_{i-1})$, $\bar{a}(-\gamma_{i-1}) = 0$ and $\bar{\pi}(t) = (\pi_1, \dots, \pi_{p-2}, \pi_p, \pi_{p+1}, \dots, \pi_t, -\pi_{p-1})$.

Lemma 3.5. Let $\sigma(y^1, \pi(t))$ be a t -simplex in $G(\gamma)$. The facet τ opposite vertex y^{t+1} lies in the boundary of $A(\gamma)$ iff

$$\pi_t = \gamma_t \text{ and } a(\gamma_t) = 0.$$

In this case we again consider three subcases:

- 1) $\gamma_t = -t$ or $\gamma_t = t$: the facet τ lies in $\text{bd } A(\gamma_t)$ and is a $(t-1)$ -simplex $\bar{\sigma}(y^1, \bar{\pi}(t-1))$ in $G(\bar{\gamma})$, where $\bar{\gamma} = (\gamma_1, \dots, \gamma_{t-1})$ and $\bar{\pi}(t-1) = (\pi_1, \dots, \pi_{t-1})$
- 2) $\gamma_t < 0$ and $\gamma_t \neq -t$: the facet τ is also a facet of the t -simplex $\bar{\sigma}(y^1, \bar{\pi}(t))$ in $G(\bar{\gamma})$, where $\bar{\gamma} = (\gamma_1, \dots, \gamma_{t-1}, -\gamma_t)$, $\bar{a}(-\gamma_t) = 0$ and $\bar{\pi}(t) = (\pi_1, \dots, \pi_{t-1}, -\pi_t)$

- 3) $\gamma_t > 0$ and $\gamma_t \neq t$: the facet τ is also a facet of the t -simplex $\sigma(y^1, \bar{\pi}(t))$ in $G(\bar{\gamma})$, where $\bar{\gamma} = (\gamma_1, \dots, \gamma_s, -\gamma_t, \gamma_{s+1}, \dots, \gamma_{t-1})$, with s such that $\gamma_s < 0$ and $\gamma_{s+1} > 0$, $\bar{a}(-\gamma_t) = a(\gamma_{s+1})$ and $\bar{\pi}(t) = (\pi_1, \dots, \pi_{p-1}, -\pi_t, \pi_p, \dots, \pi_{t-1})$ where $\pi_p = \gamma_{s+1}$.

This concludes the case that an adjacent simplex of a simplex $\sigma(y^1, \pi(t))$ in $G(\gamma)$ does not lie in $G(\gamma)$.

The algorithm now generates adjacent t -simplices $\sigma(y^1, \pi(t))$ in $G(k)$, $k=t, -t$, for varying t , $1 \leq t \leq n$, with k -complete common facets.

Definition 3.6. A g -simplex $\sigma(y^1, \dots, y^{g+1})$, $g=t-1, t$, is k -complete, $k=t, -t$, if the linear system

$$\sum_{i=1}^{g+1} \lambda_i \begin{pmatrix} z(y^i) \\ 1 \end{pmatrix} - \sum_{h=t}^{n+1} \mu_h \begin{pmatrix} e^{(h)} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.2)$$

has a solution (λ, μ) such that $\lambda_i > 0$, $i=1, \dots, g+1$, and $\mu_t > 0$ if $k=t$ and $\mu_t < 0$ if $k = -t$.

Observe that the system has $n+2$ rows and for $g=t-1$, $n+2$ columns and for $g=t$ one column more. To guarantee convergence we need the following assumption.

Nondegeneracy assumption. The linear system (3.2) has for $g=t-1$ a unique solution (λ, μ) such that $\lambda_i > 0$, $i=1, \dots, t$, and $\mu_t \neq 0$, whereas for $g=t$ at most one of the variables λ_i , $i=1, \dots, t+1$, and μ_t is equal to zero.

If \bar{z} is the piecewise linear approximation of z with respect to the underlying triangulation and $\sigma(y^1, \dots, y^{t+1})$ is a k -complete simplex, then according to (3.2) the point $x = \sum_i \lambda_i y^i$ lies in σ and satisfies

$$\begin{aligned} \bar{z}_h(x) &= 0 & h=1, \dots, t-1 \\ \bar{z}_h(x) &= \mu_h & h=t, \dots, n+1. \end{aligned} \quad (3.3)$$

A k -complete t -simplex contains a whole line segment of points satisfying (3.3). Such a line segment can be followed by making an l.p. pivot step in (3.2). At an end point either $\lambda_p = 0$ for some p or $\mu_t = 0$.

Let $\bar{C}(k)$, $k=t, -t$, be defined as the regions $C(k)$ but with respect to the function \bar{z} , then x satisfying (3.3) lies in $\bar{C}(k)$. The algorithm now follows the piecewise linear path \bar{P} in the union of $A(k) \cap \bar{C}(k)$, $k \in I_{\pm n}$, which starts at the point v . The algorithm terminates when an approximate solution to the EP has been found, i.e. when $t=n$ and μ_n becomes equal to zero. We will now describe the case that for a k -complete t -simplex $\sigma(y^1, \pi(t))$ μ_t becomes equal to zero in the linear system (3.2). If $t=n$ the point $x = \sum_{i=1}^{n+1} \lambda_i y^i$ is an approximate solution to the EP.

Lemma 3.7. Let $\sigma(y^1, \pi(t))$ be a k -complete t -simplex of $G(\gamma)$ in $A(k)$, $k=t, -t$, $1 < t < n$, with solution (λ, μ) of the linear system (3.2) such that $\mu_t = 0$. Then, if $\mu_{t+1} > 0$, $\sigma(y^1, \pi(t))$ is a facet of the $(t+1)$ -complete $(t+1)$ -simplex $\bar{\sigma}(y^1, \bar{\pi}(t+1))$ of $G(\bar{\gamma})$ in $A(t+1)$ with $\bar{\gamma} = (\gamma_1, \dots, \gamma_t, t+1)$, $\bar{a}(t+1) = 0$ and $\bar{\pi}(t+1) = (\pi_1, \dots, \pi_t, t+1)$, and if $\mu_{t+1} < 0$, then $\sigma(y^1, \pi(t))$ is a facet of the $-(t+1)$ -complete $(t+1)$ -simplex $\bar{\sigma}(y^1, \bar{\pi}(t+1))$ of $G(\bar{\gamma})$ in $A(-(t+1))$ with $\bar{\gamma} = (\gamma_1, \dots, \gamma_s, -(t+1), \gamma_{s+1}, \dots, \gamma_t)$, $\bar{a}(-(t+1)) = a(\gamma_{s+1})$ and $\bar{\pi}(t+1) = (\pi_1, \dots, \pi_{p-1}, -(t+1), \pi_p, \dots, \pi_t)$ where $\pi_p = \gamma_{s+1}$ and s is such that $\gamma_s < 0$ and $\gamma_{s+1} > 0$.

In both cases of lemma 3.7 the algorithm continues by making an l.p. pivot step with $(z^T(\bar{\gamma}), 1)$ in the linear system (3.2) where $\bar{\gamma}$ is the vertex of $\bar{\sigma}$ opposite the facet σ .

If in the case that for a k -complete t -simplex $\sigma(y^1, \pi(t))$ of $G(\gamma)$ in $A(k)$, λ_p becomes equal to zero for some p , $1 < p < t+1$, then the facet τ is either a facet of another k -complete t -simplex in $G(\gamma)$ or not. The latter case is described in the lemmas 3.3, 3.4 and 3.5 whereas the former case is described in table 1. If we consider lemma 3.3, then we observe that a k -complete facet τ cannot lie in S_h^n , for certain h , $1 < h < t-1$, since $\bar{z}_h(x)$ is positive for all x in S_h^n , so that according to (3.3) such an x cannot lie in $\bar{C}(k)$. The same holds in this case for $h=t$ and $k=-t$. If the algorithm generates a vertex y in $A(t) \cap S^n(t)$, then we set $z_t(y)$ equal to $-\epsilon$, where ϵ is some arbitrary small positive number. This prevents the algorithm from generating a t -complete facet τ in $A(t) \cap S^n(t)$. Recall from section 2 that the 2-ray process cannot generate points in $A(t) \cap S^n(t)$.

The algorithm starts with the 0-dimensional simplex $\sigma(v)$. If $z_1(v) > 0$, then v is the facet of the 1-dimensional 1-complete simplex $\sigma(y^1, (1))$ in $A(1)$ with $y^1 = v$, and if $z_1(v) < 0$, then v is the facet of the (-1) -complete 1-simplex $\sigma(y^1, (-1))$ in $A(-1)$ with $y^1 = v$. In general the algorithm generates for varying k , $k=t, -t$, $1 < t < n$, in $A(k)$ a sequence of adjacent t -dimensional k -complete simplices $\sigma(y^1, \pi(t))$ in $G(\gamma)$ for varying γ such that γ contains k . The common facet of two adjacent simplices in $A(k)$ is k -complete and the parameters $\bar{y}^1, \bar{\pi}(t)$ and \bar{a} of a t -simplex $\bar{\sigma}(\bar{y}^1, \bar{\pi}(t))$ in $A(k)$ adjacent to a t -simplex $\sigma(y^1, \pi(t))$ in $A(\gamma)$ are obtained from $y^1, \pi(t)$ and a as described in table 1 if $\bar{\sigma}$ lies also in $A(\gamma)$, and in lemma 3.4 cases 1) and 3) and lemma 3.5 cases 2) and 3) if $\bar{\sigma}$ lies in $A(\bar{\gamma})$ for some $\bar{\gamma} \neq \gamma$. In these cases the algorithm continues by making an l.p. pivot step in (3.2) with $(z^T(\bar{y}), 1)$ where \bar{y} is the vertex of $\bar{\sigma}$ opposite the common facet τ . If the k -complete facet τ lies in the boundary of $A(k)$, i.e. when case 2) of lemma 3.4 or case 1) of lemma 3.5 occurs, then τ is a $(t-1)$ -simplex $\sigma(y^1, \bar{\pi}(t-1))$ in either $A(t-1)$ or $A(-(t-1))$ and the algorithm continues by

reintroducing $e(t-1)$ in (3.2) by either increasing μ_{t-1} or decreasing μ_{t-1} from zero. If the algorithm generates a solution (λ, μ) of the linear system (3.2) with $\mu_t = 0$, then in the case $t=n$ the point x given by $x = \sum_{i=1}^{n+1} \lambda_i y^i$ is an approximate solution of the EP, and in the case $t < n$, σ is a $(t+1)$ -complete t -simplex in $A(t+1)$ if $\mu_{t+1} > 0$ and a $-(t+1)$ -complete t -simplex in $A(-(t+1))$ if $\mu_{t+1} < 0$. The parameters of the unique $(t+1)$ -simplex $\bar{\sigma}(y^1, \bar{\pi}(t+1))$ in $A(k)$, $k=t+1$ or $-(t+1)$, containing σ as facet, are obtained as described in lemma 3.7.

By the description given above the algorithm traces a piecewise linear path \bar{P} in $\bar{B} = \cup_k A(k) \cap \bar{C}(k)$ where the union is over all k in $I_{\pm n}$. The path \bar{P} connects v with an approximate solution x of the EP. Notice that according to (3.3) the path \bar{P} can be considered as an approximation of the piecewise smooth path P in B . If the accuracy of the approximate solution x is not satisfactory, the algorithm can be restarted with v equal to x and with a smaller grid size of the triangulation.

4. Concluding remarks

As described in section 3 the 2-ray algorithm solves a sequence of subproblems $\bar{z}_1(x) = 0, \dots, \bar{z}_t(x) = 0$ for varying t , $1 \leq t \leq n$. However, it is also possible to solve the sequence of subproblems $\bar{z}_{i_1}(x) = 0, \dots, \bar{z}_{i_t}(x) = 0$, where (i_1, \dots, i_t) is a (fixed) permutation of t elements in I_{n+1} . Once a permutation is chosen it remains fixed during the remaining part of the algorithm. If an index i_{t+1} has to be added, then we can choose i_{t+1} in such a way that $|\mu_{i_{t+1}}| = \max\{|\mu_h| \mid h \in I_{n+1} \setminus \{i_1, \dots, i_t\}\}$. For the starting point $x=v$ this coincides with choosing i_1 in such a way that $|\bar{z}_{i_1}(v)| = \max_{h \in I_{n+1}} |\bar{z}_h(v)|$. The regions $A(k)$, $k=t, -t$, are adapted such that the permutation $(1, \dots, t)$ is replaced by (i_1, \dots, i_t) , $t=1, \dots, n$.

The 2-ray algorithm for solving the equilibrium problem can be easily adapted to solve the more general nonlinear complementarity problem (NLCP) on S^n which can be stated as follows. Let z be a continuous function from S^n into R^{n+1} such that $x^T z(x) = 0$ for all x in S^n , then the NLCP is to find an x^* in S^n such that $z(x^*) < 0$. In this case a starting point can lie on the boundary of S^n and we must allow for movements on the boundary.

Algorithms discussed on S^n can be generalized to the product space S of N , $N > 1$, unit simplices S^{n_j} , $j=1, \dots, N$. A generalization of the 2-ray algorithm would then be the 2^N -ray algorithm on S which can be applied to the EP and the NLCP on S . Again we must take into account that the starting point can lie on the boundary of S and we must allow for movements on the boundary.

The 2^N -ray algorithm on $S = \prod_{j=1}^N S^{n_j}$, $N > 1$, to solve the NLCP on S will be discussed in a subsequent paper.

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